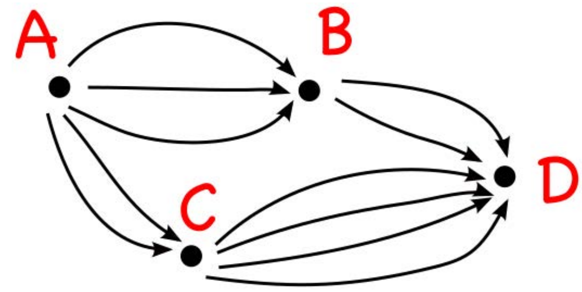
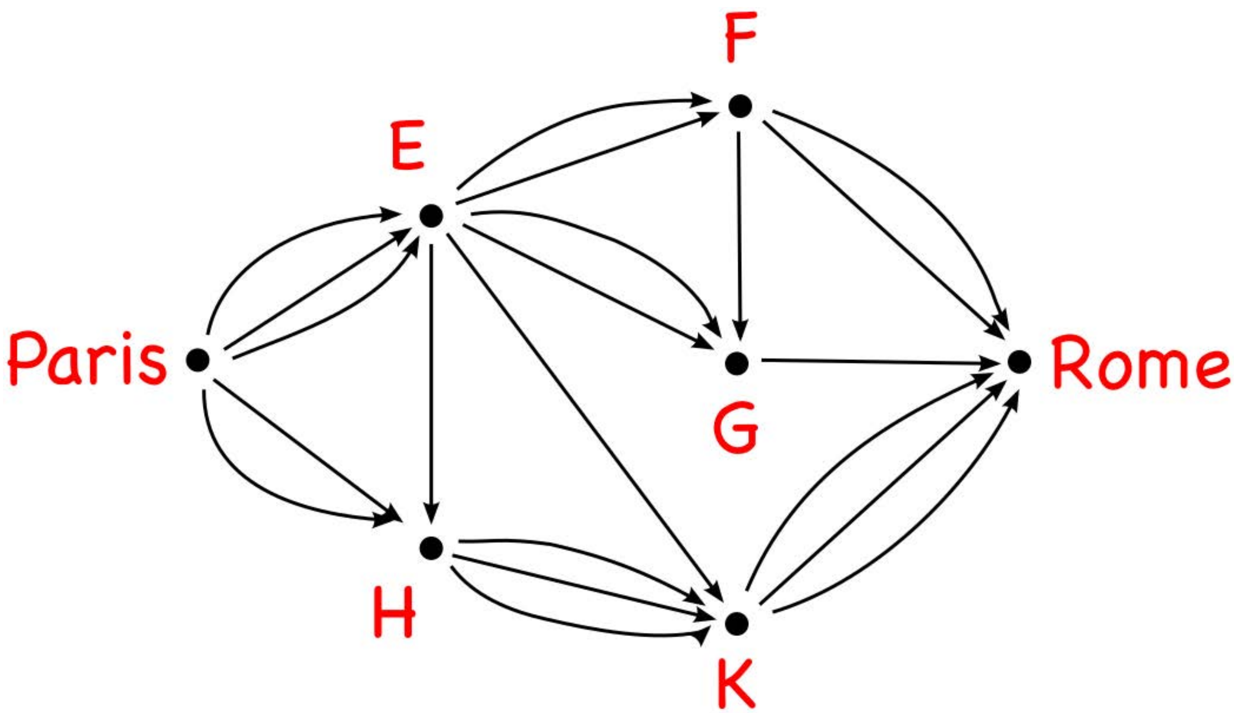
Puzzle of the Week

All Roads Lead to Rome – 2

This map shows a collection of one-way roads going between cities A, B, C, and D. To get from A to D you can either go through B or C. There are 3 x 2 = 6 ways of going from A to D if you go through B. There are 2 x 4 = 8 ways of going from A to D if you go through C. Therefore, there are a total of 6 + 8 = 14 ways of going from A to D.



THE CHALLENGE: Here is a complicated pretend map of one-way roads leading from Paris to Rome. How many possible routes are there?



EXPLORATION: Why does this puzzle not work using two-way roads? Make some fun maps for your friends to puzzle over.

Puzzle of the Week

All Roads Lead to Rome – 2 – Notes

THE CHALLENGE: This puzzle involves several important counting methods.

- If you have independent ways to do two steps of something, the total number of ways is the product of the number of ways of doing each of the steps..
- If two things are completely separate, then their ways add up.
- You can sometimes count things by flooding out from a beginning point.

Let's start in Paris and make our way to Rome.

- E - There are 3 ways to get to E from Paris.
- H - There are 2 ways to get here directly from Paris, and 3 x 1 = 3 ways of getting to H from E. Therefore, there are a total of 2 + 3 = 5 ways of getting to H from Paris.
- F - 3 ways to go from Paris to E; 2 ways to get from E to F; so 3 x 2 = 6 ways from Paris to F.
- G - There are two kinds of routes for getting to G - either directly from E or from F. There are 3 x 2 ways of getting to G straight from E. There are 6 x 1 = 6 ways of getting to G from F. Thus, there are a total of 6 + 6 = 12 ways of getting to G.
- K - You can get to K either from H or E. Going through H there are 5 x 3 = 15 ways. Going through E there are 3 x 1 = 3 ways. Altogether there are 15 + 3 = 18 ways of getting to K.
- Rome - The last step of going to Rome can be through F, G, or K. Through F there are 6 x 2 = 12 ways. Through G there are 12 x 1 = 12 ways. Through K there are 18 x 3 = 54 ways. That makes a grand total of 12 + 12 + 54 = 78 ways of going from Paris to Rome.

EXPLORATION: If you have two-way roads, that would create loops that would create an infinite number of possible routes.

Puzzle of the Week

Boxed Blocks

THE CHALLENGE: There is a wooden box, without a lid, holding a 4 x 4 x 4 collection of 64 blocks. How many of the blocks touch some part of the box?

EXPLORATION: What happens for a 5 x 5 x 5 collection of 125 blocks in a bigger box? What about other sizes of boxes? How do these answers change if the boxes have a lid that touches the top row of blocks?

Puzzle of the Week
Boxed Blocks – Notes

THE CHALLENGE: There are different ways to go about solving this problem.

1. Think of the 2 x 2 x 3 = 12 core of blocks that don't touch the sides.
2. If it had a lid, there would be 2 x 2 x 2 = 8 blocks not touching the box. In addition to those 8, there would be 2 x 2 x 1 = 4 blocks only touching the lid.
3. You can add up the blocks along the sides and subtract them from the 4 x 4 x 4 = 64 total. This is the hardest way. There are 4 x 4 x 1 = 16 blocks on the bottom of the box. There are 3 x 12 = 36 blocks around the sides that don't touch the bottom. Therefore, there are 64 - (16 + 36) = 12 blocks on the inside.

EXPLORATION: Look at these only using the first method.

5 x 5 x 5. Without a lid, there are 3 x 3 x 4 = 36 blocks not touching the box. With a lid, there are 3 x 3 x 3 = 27 blocks not touching.

a x b x c, where c is the height. Assume these numbers are each at least three. Without a lid, a and b are reduced by 2 and c by 1, so there are (a - 2) x (b - 2) x (c - 1) blocks. With a lid, (a - 2) x (b - 2) x (c - 2) blocks not touching.

Puzzle of the Week

Extreme Products – 1

THE CHALLENGE: 1) Using the digits from 1 to 9, each at most once, make two 2-digit numbers whose product is as large as possible. 2) Also, using the digits from 1 to 9, each at most once, make two 2-digit numbers whose product is as small as possible.

□ □ × □ □

1 2 3 4 5 6 7 8 9

EXPLORATION: Can you apply what you learned for multiplying two numbers to do this with multiplying three two-digit numbers? Can you think of other interesting variations?

Puzzle of the Week

Extreme Products – 1 – Notes

THE CHALLENGE: This is small enough and obvious enough, that not much experimenting or analysis is needed.

To make the product large we want to have tens digits that are 8 and 9. The only remaining question can be settled by multiplying: should it be 97 x 86 = 8342 or 96 x 87 = 8352.

Similar logic for making the product small produces a choice of 13 x 24 = 312 or 14 x 23 = 322.

EXPLORATION: For three numbers, the analysis is similar.

To make the product large, we have first digits of 7, 8, and 9, and second digits of 4, 5, and 6. The only question is how to combine them. Some experimentation verifies what you would expect from the two number case: the largest value comes from 94 x 85 x 76.

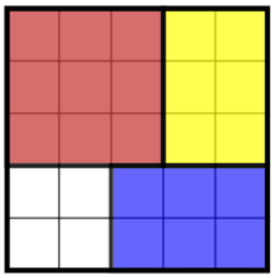
Similarly, the smallest value comes from 14 x 25 x 36.

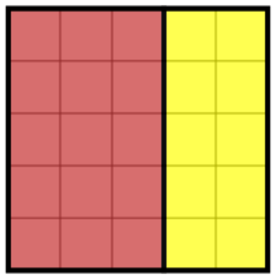
A natural variation of this problem is to look at three-digit numbers. We will do this in the "Extreme Products - 2" puzzle.

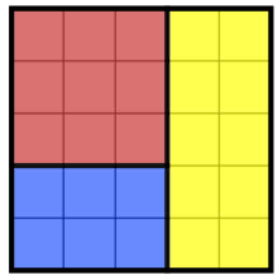
Puzzle of the Week

Filling Squares with Different Rectangles

The goal is to fill a square with all different rectangles that are roughly the same size. Use the score, which is the difference of the areas of the largest and smallest rectangles, to measure how successful the diagram is. The scores of these three squares, from left to right, are 9 - 4 = 5, 15 - 10 = 5, and 10 - 6 = 4. However, the leftmost square is not allowed as it has two rectangles with the same dimensions (2 by 3 and 3 by 2).







THE CHALLENGE: Find designs with the lowest scores you can for legally breaking up a 3 by 3, 4 by 4, 5 by 5, and 6 by 6 square.

EXPLORATION: Continue your exploration with 7 by 7, 8 by 8, 9 by 9, and 10 by 10 squares.

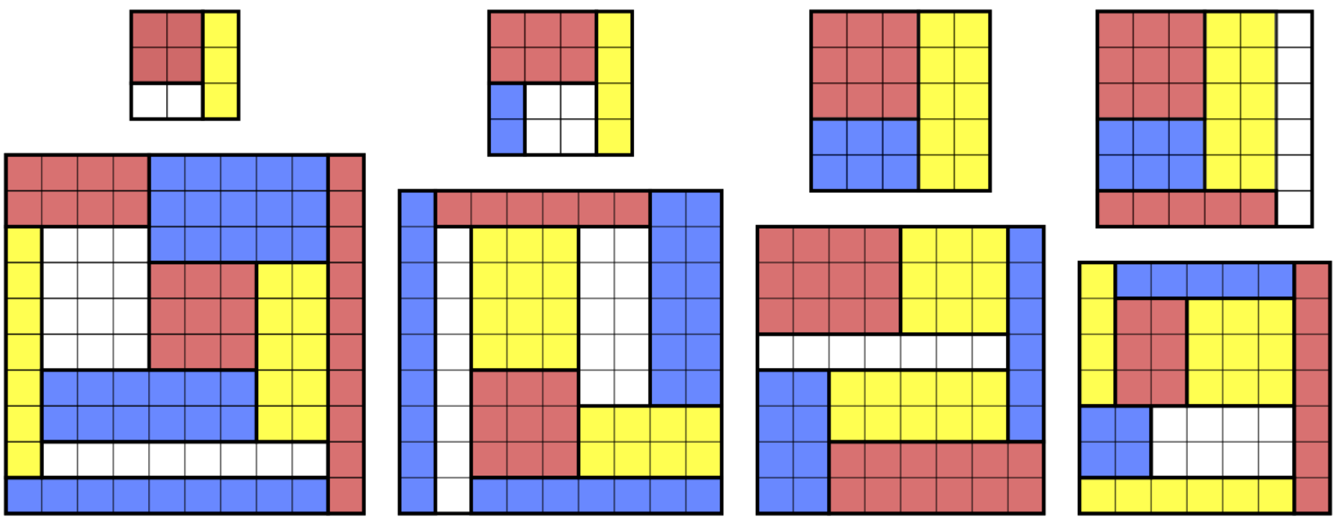
Puzzle of the Week

Filling Squares w/ Different Boxes – Notes

THE CHALLENGE & EXPLORATION: Here are some solutions for 3 by 3 through 10 by 10. There are of course other solutions, not shown, for most of these sizes.

The difference in areas is:

- 3: 4 - 2 = 2
- 4: 6 - 2 = 4
- 5: 10 - 6 = 4
- 6: 10 - 5 = 5
- 7: 9 - 4 = 5
- 8: 12 - 6 = 6
- 9: 12 - 6 = 6
- 10: 15 - 7 = 8



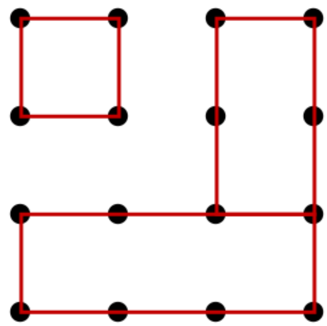
There are some general principles that can help get you started. When looking at a square with odd dimensions, such as 7 by 7, it can always be broken up nearly in half as a 4 by 7 and a 3 by 7 - this has a score of 7. In general, this method provides a score of n for an n by n square with n odd. This is not usually the best possible score, as is seen by the 5 by 5 example on the main page, yet it is a good place to start.

Another relatively easy thing to try is to put a narrow strip (one or two wide) on the right and bottom edges together with a previous solution. This type of construction was used for the 3 by 3, 5 by 5, 6 by 6, 7 by 7, and 10 by 10.

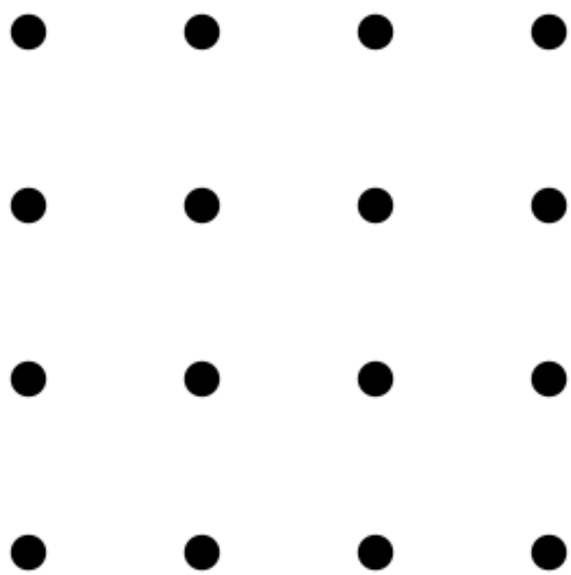
Puzzle of the Week

Finding Rectangles in Squares

Drawn in red in this grid are a 1 by 1 rectangle (square), a 2 by 1 rectangle and a 1 by 3 rectangle.



THE CHALLENGE: Count the number of all the rectangles you can make in this 4 by 4 grid.



EXPLORATION: Calculate the number of rectangles in a 2 by 2 grid and a 3 by 3 grid. Compare those answers to your answer for the 4 by 4 grid. What is the pattern you see, and can you explain it?

Puzzle of the Week

Finding Rectangles in Squares – Notes

THE CHALLENGE & EXPLORATION: The careful student will find 36 rectangles. The key is to be organized in your approach. Here is a count for each type of rectangle:

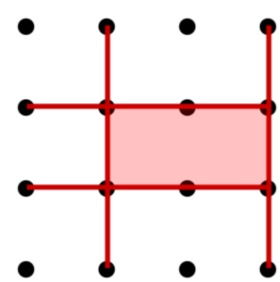
- 1 x 1 - 9
- 1 x 2 - 6
- 1 x 3 - 3
- 2 x 1 - 6
- 2 x 2 - 4
- 2 x 3 - 2
- 3 x 1 - 3
- 3 x 2 - 2
- 3 x 3 - 1

Adding these all together, we get 9 + 6 + 3 + 6 + 4 + 2 + 3 + 2 + 1 = 36 rectangles.

For n by n grids, the number of rectangles is:

- 2 by 2 - 1 = 1^2
- 3 by 3 - 9 = 3^2
- 4 by 4 - 36 = 6^2
- 5 by 5 - 100 = 10^2
- 6 by 6 - 225 = 15^2

They are all squares. What are the numbers 1, 3, 6, 10, and 15? They are the triangular numbers, and they are also the number of ways of choosing two things from n things.



To see why this works, consider how a rectangle is determined inside the square. As shown in this illustration, every rectangle is determined by picking the two vertical sides and the two horizontal sides. And for each choice of positions for the sides, there is one rectangle that is determined.

So, the number of rectangles is equal to the number of ways of making these choices. In a 4 by 4 grid, there are 4 vertical lines to choose from and 4 horizontal lines to choose from. The number of ways of choosing 2 things from 4 things is 6. So the number of rectangles is 6 x 6 = 36.

The same reasoning is true for other sizes of square grids.

Puzzle of the Week

Letter Substitutions – 10

Rules:

1. A letter represents a digit from 0 to 9, and has the same value throughout a single puzzle.
2. No number can start with the digit 0.
3. Within a puzzle, different letters must have different values.

$$\begin{array}{r} 8 \\ \underline{+\ \ A} \\ B\ \ 2 \end{array} \quad \Rightarrow \quad \begin{array}{r} 8 \\ \underline{+\ \ 4} \\ 1\ \ 2 \end{array}$$

THE CHALLENGE: Find the value of S, E, N, D, M, O, R, and Y to make this puzzle work.

$$\begin{array}{r} S\ E\ N\ D \\ \underline{+\ M\ O\ R\ E} \\ M\ O\ N\ E\ Y \end{array}$$

EXPLORATION: Make some letter substitution puzzles for your friends to solve.

Puzzle of the Week

Letter Substitutions – 10 – Notes

THE CHALLENGE: Because carries when adding two digits, plus a possible carry, are at most 1, the first thing to notice is that M must be 1.

Now that M is 1, look at S + 1 + (possible carry) = 1O on the left side of the puzzle. S + 1 + (possible carry) cannot be more than 11, and since letters cannot repeat another letter's value, O must be 0.

Our puzzle now looks like this:

```
  S E N D
+ 1 0 R E
---------
1 0 N E Y
```

Because E + 0 = N, there must be a carry into that column and N must be one more than E. Also, that column does not produce a carry into the next column. This means S + 1 = 10, so S = 9.

With N being one more than E, and N + R + (possible carry) = 1E, this forces R = 8 and there is indeed a carry into that column.

Our puzzle now looks like this:

```
  9 E N D
+ 1 0 8 E
---------
1 0 N E Y
```

At this point there is nothing clever to do other than look at possibilities while remembering that E and N are consecutive numbers. The only way D + E = 1Y without duplicating values is for D = 7, E = 5, Y = 2, and N = 6.

At last we have the solution:

```
  9 5 6 7
+ 1 0 8 5
---------
1 0 6 5 2
```

Notice how helpful it is to update the puzzle each time we determine the value of some new letters.

Puzzle of the Week

Maximizing Products with 16

Look at ways to break up 16 into a sum of numbers that you can then multiply to get as big a product as possible. Writing 16 = 10 + 6 is a start, but 16 = 6 + 5 + 5 is better. Can you do better?

$$16 = 10 + 6 \text{ and } 10 \times 6 = 60$$

$$16 = 6 + 5 + 5 \text{ and } 6 \times 5 \times 5 = 150$$

THE CHALLENGE: What is the biggest product you can make by breaking 16 into a sum of numbers?

EXPLORATION: How does your strategy change if you replace 16 with 20, 50, or 100?

Puzzle of the Week

Maximizing Products with 16 – Notes

THE CHALLENGE & EXPLORATION: Let's start by looking at some examples that replace 16 with smaller numbers. Getting this experience is almost always a good way to start, and it usually doesn't take long.

1 = 1 => 1

2 = 2 => 2

3 = 3 => 3

4 = 2 + 2 or just 4 => 4

5 = 2 + 3 => 2 x 3 = 6

6 = 3 + 3 => 3 x 3 = 9

7 = 2 + 2 + 3 => 2 x 2 x 3 = 12

8 = 2 + 3 + 3 => 2 x 3 x 3 = 18

9 = 3 + 3 + 3 => 3 x 3 x 3 = 27

10 = 2 + 2 + 3 + 3 => 2 x 2 x 3 x 3 = 36

As you look at these examples, a pattern emerges for breaking down larger numbers.

1. Any number larger than 4 should be replaced with smaller numbers!
2. Never use 1 unless you have no choice.
3. Use 2 + 2 instead of 4. They give the same result, but it is easier to see how to improve things when looking at 2's rather than 4's.
4. Always replace 2 + 2 + 2, which gives 2 x 2 x 2 = 8, with 3 + 3, which gives 3 x 3 = 9.

Using these rules, a general strategy emerges for attacking any number.

1. If the number is even, write it as a sum of 2's. If the number is odd, write it as 3 plus a sum of 2's.
2. Replace any group of three 2's with a group of two 3's.

With all this in place, we are ready to solve the problem.

16 = 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 = 2 + 2 + 3 + 3 + 3 + 3, which gives a product of 2 x 2 x 3 x 3 x 3 x 3= 324.

As for the other numbers, we have:

20 = 10 x 2 = 2 + 3 x (3 x 2) = 2 + 3 x (2 x 3) = 2 + 6 x 3, which gives 2 x 3^6 = 2 x 729 = 1458.

50 = 25 x 2 = 2 + 8 x (3 x 2) = 2 + 8 x (2 x 3) = 2 + 16 x 3, which gives 2 x 3^16.

100 = 50 x 2 = 2 x 2 x 16 x (3 x 2) = 2 + 2 + 16 x (2 x 3) = 2 x 2 + 32 x 3, which gives 4 x 3^32.

Puzzle of the Week
Moving Digits – 2

THE CHALLENGE: Find a 4-digit number, ABCD, that satisfies this interesting equation when you reverse the digits:

$$ABCD \times 4 = DCBA$$

EXPLORATION: Investigate why this cannot happen for numbers less than 1000. Also, look for numbers larger than 9,999 that have this property.

Puzzle of the Week

Moving Digits – 2 – Notes

THE CHALLENGE: Because multiplying ABCD by 4 produces a 4-digit number, A must be 1 or 2. Also, because the result of multiplying D by 4 has a ones digit of A, we know A must be even. So A = 2.

Our equation is now 2BCD x 4 = DCB2.

D x 4 produces a 2 means that D is 3 or 8. Notice that 4 times a number larger than 2000 creates a number that is at least 8000. Hence, D must be 8.

Our equation is now 2BC8 x 4 = 8CB2.

B must be less than 3, or 2B x 4 will be bigger than 8999. Looking at C8 x 4 and going through the ten values of C, the only way to get a value of B in that range is if C is 2 or 7 and B is 1. Therefore, we only have two numbers to check: 2128 or 2178.

The answer is 2178!

EXPLORATION: The logic that shows that the number must have a high-order digit of 2 and a low-order digit of 8 holds no matter how many digits the number has.

Looking at two-digit numbers, 28 does not work. For three-digit numbers, 2x8 does not work for any value of x.

The analysis for larger numbers is perhaps more than you'd like to read. Here are the next few numbers: 21978, 219978, and 2199978.

Puzzle of the Week

Moving Digits – 3

THE CHALLENGE: Find a 4-digit number, ABCD, that satisfies this interesting equation when you reverse the digits:

$$ABCD \times 9 = DCBA$$

EXPLORATION: Investigate why this cannot happen for numbers less than 1000. Also, look for numbers larger than 9,999 that have this property.

Puzzle of the Week

Moving Digits – 3 – Notes

THE CHALLENGE: Because multiplying ABCD by 9 produces a 4-digit number, A must be 1.

Our equation is now 1BCD x 9 = DCB1.

D x 9 produces a 1 means that D must be 9.

Our equation is now 1BC9 x 9 = 9CB1.

B must be less than 2 so that 1BC9 x 9 will be less than 9999. C9 x 9 ends in B1 leaves only two possibilities for C - either C is 8 (B = 0) or 7 (B = 1). This gives two possible numbers: 1089 or 1179. Only 1089 works.

The answer is 1089.

EXPLORATION: No matter the number of digits, the logic holds that the high-order digit should be 1 and the low-order digit should be 9.

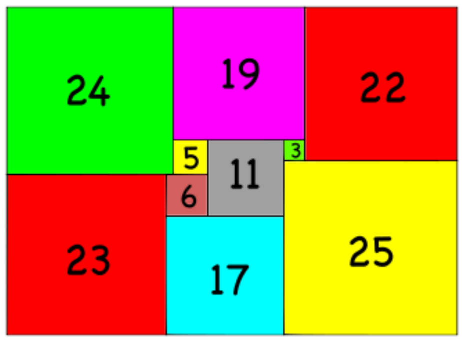
For two-digit numbers, 19 does not work. For three-digit numbers, numbers of the form 1x9 do not work.

The analysis for larger numbers is perhaps more than you'd like to read. Here are the next few numbers beyond 9999: 10989, 109989, and 1099989.

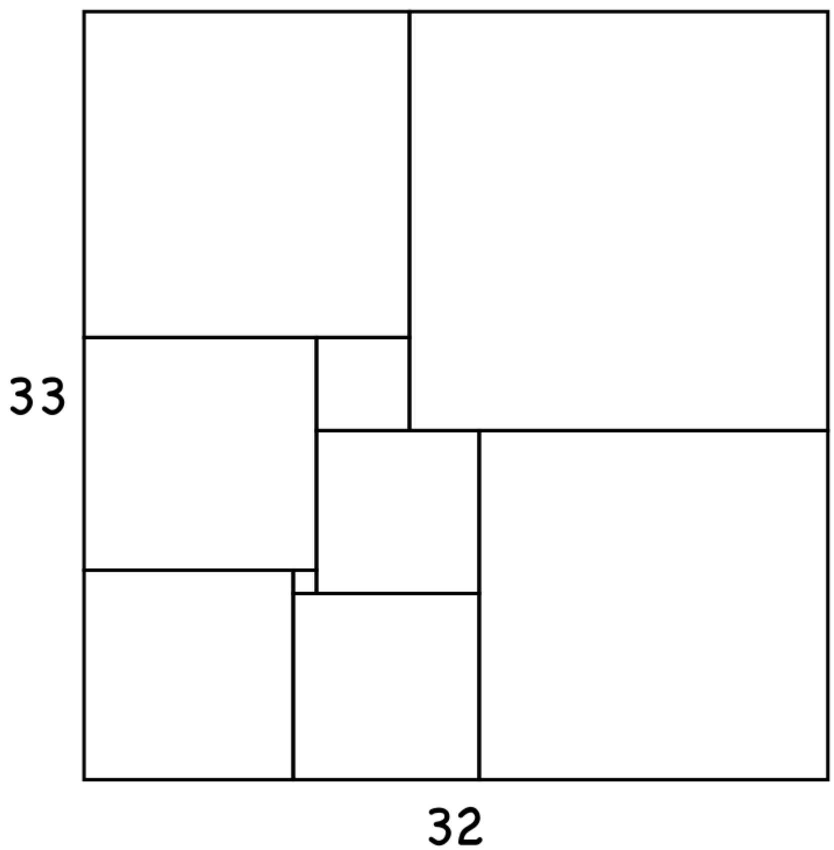
Puzzle of the Week

Perfect Rectangles – 1

A rectangle that can be filled with squares, no two of which have the same size, is called a ***Perfect Rectangle***. Here is an example of a 47 by 65 Perfect Rectangle filled with 10 squares of different sizes.



THE CHALLENGE: Find the sizes of the squares in this 33 by 32 Perfect Rectangle filled with 9 squares.



Puzzle of the Week

Perfect Rectangles – 1 – Notes

THE CHALLENGE: Your students can measure the diagram carefully to come up with the sizes, but hopefully they will enjoy the challenge of doing the problem solving instead.

A good place to start is with the six squares in the lower left corner.

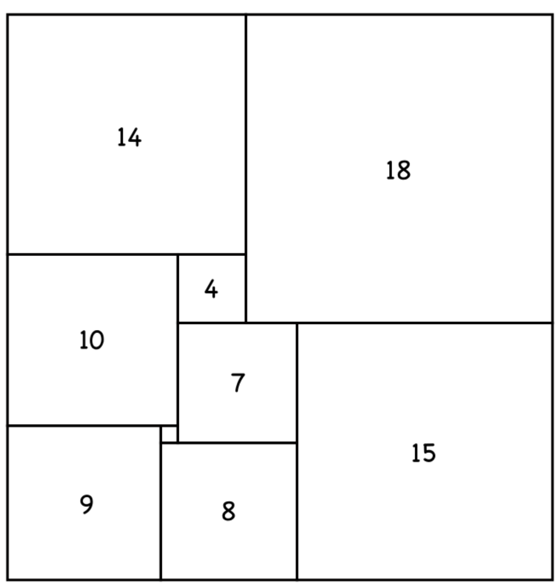
Let n be the size of the smallest square, and x the size of the square in the Middle of the Bottom row (call it MB for short).

The squares that surround the smallest square increase in size by n as they go around the smallest square. The square on top of MB has size x - n, and the square in the bottom left corner has size x + n. Also the square in the bottom right corner has the combined sizes of MB and the square on top of MB, so its size is x + (x - n) = 2x - n.

The sizes of the squares on the bottom “row” of this diagram add up to 32, so 32 = (x + n) + x + (2x - n) = 4x, which means x = 8.

The stack along the bottom left side is (x + n) + (x + 2n) = 2x + 3n high. The stack of three squares in the bottom middle is x + (x -n) + middle square = 2x - n + middle square. Because 2x + 3n = 2x - n + middle square, the middle square is 4n in size. If n is 2, we would have two squares of size 8. If n is 3 or bigger, that middle square would be too big. So, n = 1.

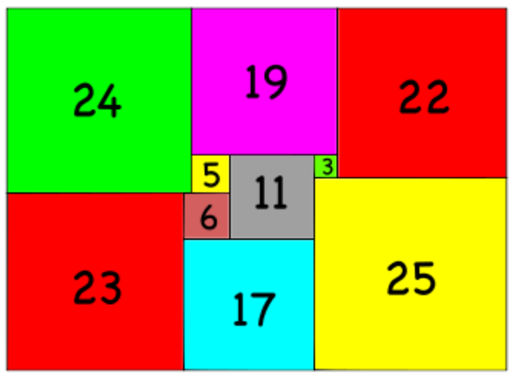
With x = 8 and n = 1, and knowing the size of the rectangle as 33 by 32, filling in the rest of the squares is automatic.



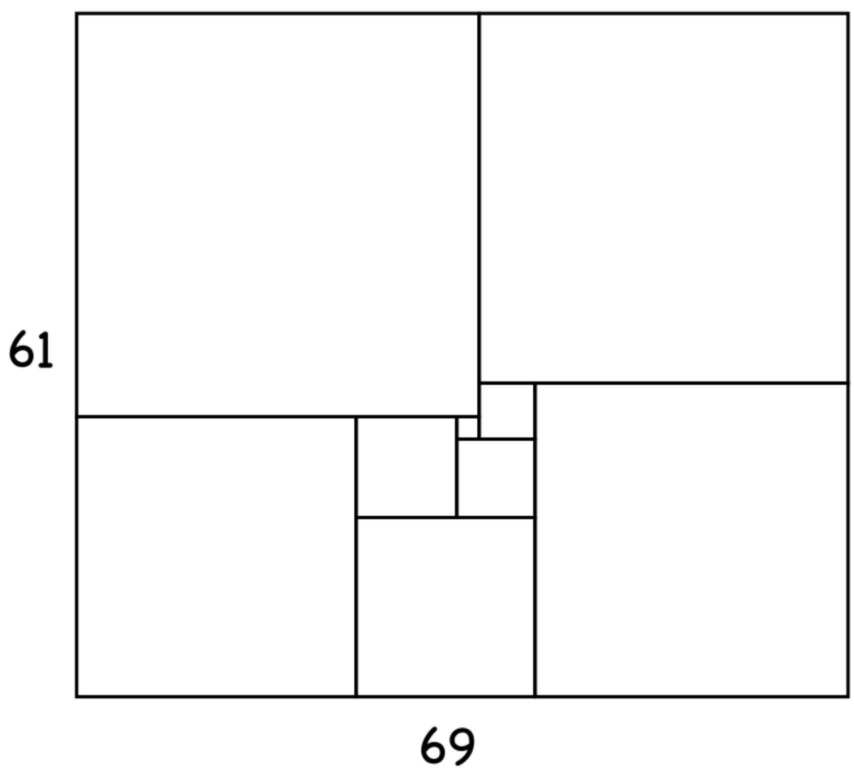
Puzzle of the Week

Perfect Rectangles – 2

A rectangle that can be filled with squares, no two of which have the same size, is called a ***Perfect Rectangle***. Here is an example of a 47 by 65 Perfect Rectangle filled with 10 squares of different sizes.



THE CHALLENGE: Find the sizes of the squares in this 61 by 69 Perfect Rectangle filled with 9 squares.



Puzzle of the Week

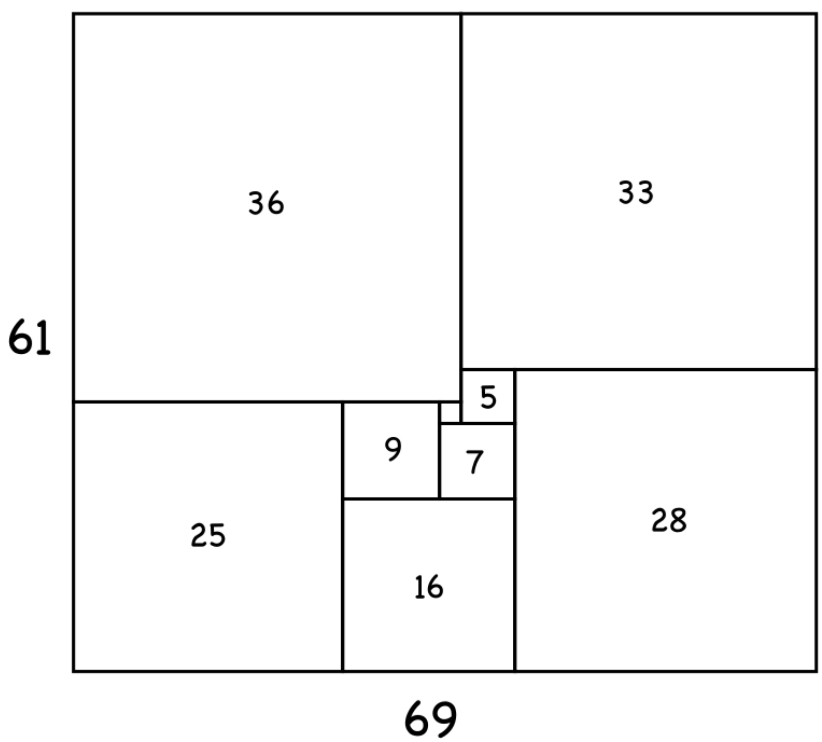
Perfect Rectangles – 2 – Notes

THE CHALLENGE: One of the easier places to start is to look at the four large squares in the four corners.

The size of the upper left plus the upper right is 69, while the upper left plus the lower left is 61, so that means the lower left is 8 less than the upper right. Similarly, the size of the upper right plus upper left is 69, while the upper right plus the lower right is 61, so that means the lower right is 8 less than the upper left. Putting that together, the sum of the two upper squares is 16 more than the sum of the two bottom corners. Therefore, the square in the middle of the bottom must have size 16.

The two squares on top of the size 16 square have sizes that add up to 16 and a difference that is the size of the central small square. Therefore, the central square has size 2, 4, or 6. However, 4 and 6 are visibly too large, so the central squares size is 2, and the two squares on top of the 16 square are 7 and 9.

The rest of the sizes follows pretty easily from that information and knowing the size of the rectangle.

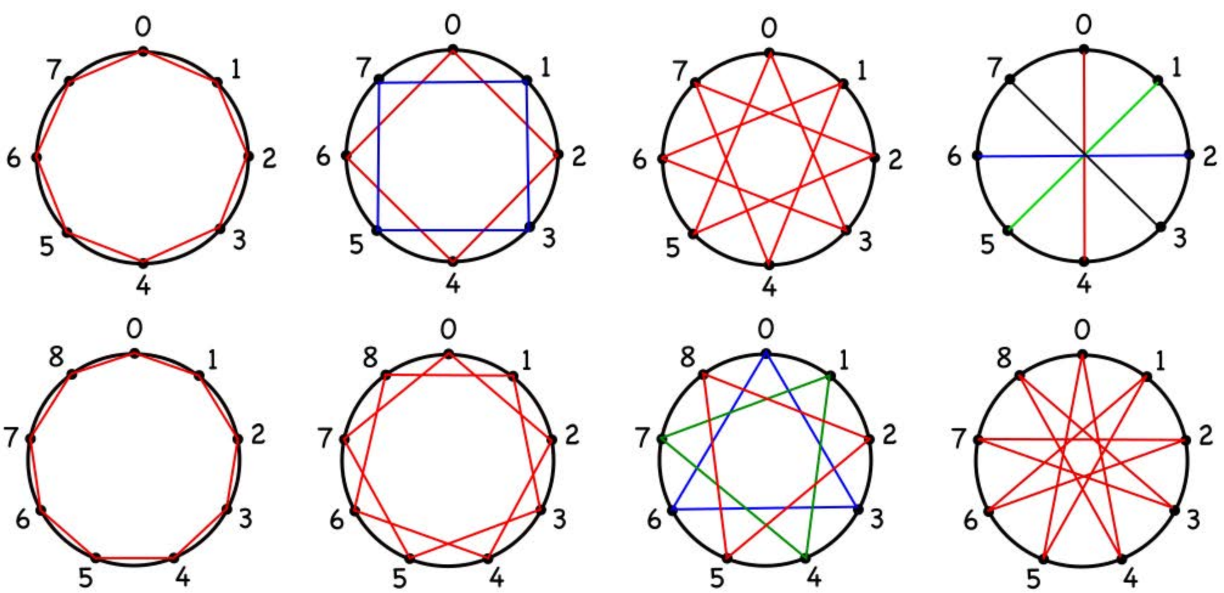


Puzzle of the Week

Stars and Loops Inside Circles

Place some pins evenly spaced around a circle. Picture below are circles with 8 and 9 pins. Make a star by setting a skip amount and attaching one color of string from pin to pin using that skip amount. For example, the third picture of the first row skips by 3 each time, and connects 0 - 3 - 6 - 1 - 4 - 7 - 2 - 5 - 0. In that example, one loop of string connects all the numbers. In other examples, several loops of string were needed to involve all the numbers.

THE CHALLENGE: Suppose you have 12 pins evenly spaced around a circle. For each skip amount, find how many loops of string you will need and how many numbers will be in each loop of string. What do you notice about the loop lengths?



EXPLORATION: Investigate what happens for other numbers of pins. Can you predict what will happen with the loop lengths and number of loops for a given number of pins and skip amount?

Puzzle of the Week

Stars and Loops Inside Circles – Notes

THE CHALLENGE: For 12 pins, the following occurs for the different skip amounts:

Skip Amount	1	2	3	4	5	6	7	8	9	10	11
Number of Loops	1	2	3	4	1	6	1	4	3	2	1
Loop Length	12	6	4	3	12	2	12	3	4	6	12
GCD of 12, Skip	1	2	3	4	1	6	1	4	3	2	1

Before looking at the table, it is worth noticing that the loop length is the same for all the loops for a given skip amount.

A few things stand out from this table. Perhaps the easiest is that the number of loops times the loop length is always the number of pins. Another is that it is symmetric about the “6” column.

If you look at the GCD (or GCF) of the number of pins (12) and the skip amount, you’ll notice that it’s always the same as the number of loops!

EXPLORATION: The observations from The Challenge should be pursued.

1. For a given skip amount and pin count, why are all the loops the same length? Begin with the loop that starts at 0 and ends at 0. Every number in the loop that starts at 1 will be 1 more than the ones in the loop that starts at 0, and so the loop that starts at 1 will have the same length. The same logic holds for any starting point.

2. For a given skip amount and pin count, why is the product of the number of loops and the loop length equal to the number of pins? The loops cannot overlap and they include all the numbers, so this result follows easily.

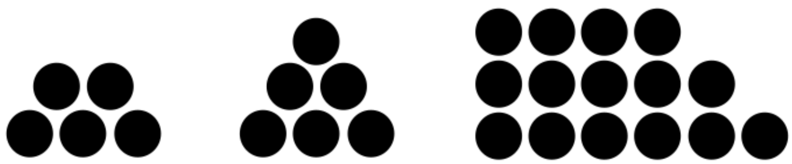
3. Why is the GCD of the skip amount and the pin count equal to the number of loops? First notice that all the numbers in the loop that starts at 0 are exactly the multiples of the GCD. After that, the discussion of point (1) makes it clear that there will be that many loops before the loops start repeating their content.

4. Why is the table symmetric about the middle column? This symmetry is due to the GCD being symmetric. GCD(n, k) = GCD(n, n - k), where n is the number of pins and k is the skip amount.

Puzzle of the Week

Trapezoidal Numbers – 2

Trapezoidal Numbers are the sum of two or more consecutive numbers. They deserve their name because you can make a trapezoid with that many dots, as pictured in the examples below. Note that having 1 dot on the top row is stretching the idea of being a trapezoid a bit, but it is allowed for these numbers.



5 6 15

THE CHALLENGE: 100 is a Square Number and a Trapezoidal Number. Find a group of consecutive numbers that add up to 100.

100 =

EXPLORATION: Can you find other ways of adding up consecutive numbers to get a sum of 100?

Puzzle of the Week

Trapezoidal Numbers – 2 – Notes

THE CHALLENGE & EXPLORATION: Because consecutive numbers form an arithmetic sequence, the formula for the sum of k consecutive numbers starting at n is k x (n + (n + k - 1)) / 2. You can think of this as k times the average value of all the numbers. You can also think of it as k times the median value of this arithmetic sequence of numbers.

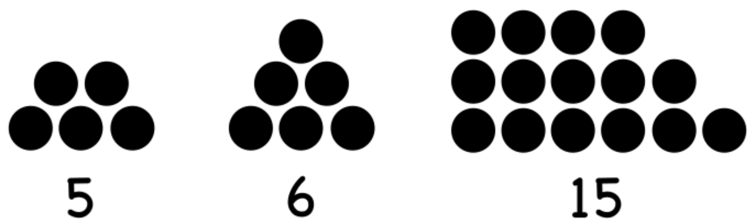
If 100 = k x (n + (n + k - 1)) / 2, then 200 = k x (2n + k - 1). Each way of factoring 200 as a smaller number times a larger number gives us a possibility to consider:

- 1 x 200: k = 1, n = 100. We need at least 2 numbers, so k = 1 is impossible.
- 2 x 100: k = 2, n = 49 ½. n must be an integer, so this doesn't work.
- 4 x 50: k = 4, n = 23 ½. n must be an integer, so this doesn't work.
- 5 x 40: k = 5, n = 18. 100 = 18 + 19 + 20 + 21 + 22. Bingo!
- 8 x 25: k = 8, n = 9 ½. n must be an integer, so this doesn't work.
- 10 x 20: k = 10, n = 5 ½. n must be an integer, so this doesn't work.

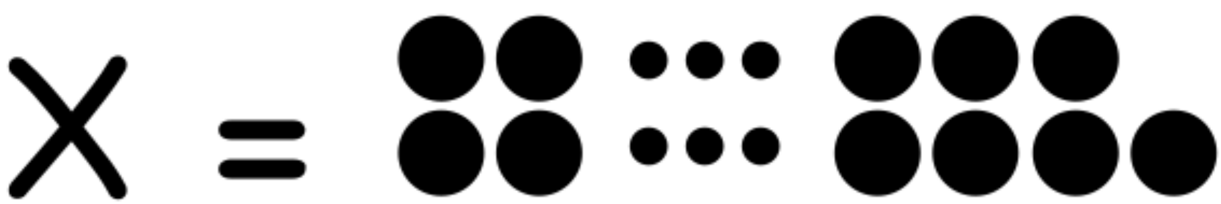
So, 100 = 18 + 19 + 20 + 21 + 22 is the only way!

Puzzle of the Week
Trapezoidal Numbers – 3

Trapezoidal Numbers are the sum of two or more consecutive numbers. They deserve their name because you can make a trapezoid with that many dots, as pictured in the examples below. Note that having 1 dot on the top row is stretching the idea of being a trapezoid a bit, but it is allowed for these numbers.



THE CHALLENGE: Which numbers can be expressed as the sum of 2 consecutive numbers?



EXPLORATION: Are there easy ways to describe numbers that can be expressed as the sum of 3 consecutive numbers? 4 numbers? 5 numbers?

Puzzle of the Week

Trapezoidal Numbers – 3 – Notes

THE CHALLENGE & EXPLORATION: If a number is the sum of two consecutive numbers, then it is equal to n + (n + 1), which is 2n + 1. Numbers of the form 2n + 1 are the odd numbers starting with 3.

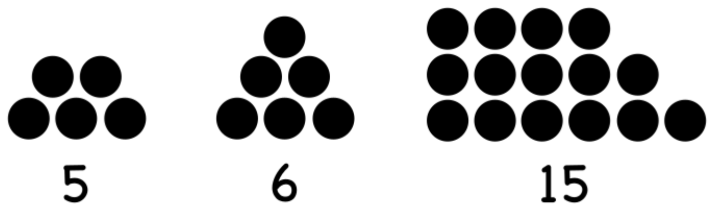
The sum of three consecutive numbers is (n - 1) + n + (n + 1) = 3n. Any multiple of 3 starting with 6 will be the sum of three consecutive numbers.

The sum of four consecutive numbers is (n - 1) + n + (n + 1) + (n + 2) = 4n + 2 = 2(2n + 1). Any number that is twice an odd number, starting with 10, will be the sum of four consecutive numbers.

The sum of five consecutive numbers is (n - 2) + (n - 1) + n + (n + 1) + (n + 2) = 5n. Any multiple of 5 starting with 15 will be the sum of five consecutive numbers.

Puzzle of the Week
Trapezoidal Numbers – 4

Trapezoidal Numbers are the sum of two or more consecutive numbers. They deserve their name because you can make a trapezoid with that many dots, as pictured in the examples below. Note that having 1 dot on the top row is stretching the idea of being a trapezoid a bit, but it is allowed for these numbers.



THE CHALLENGE: Which numbers can be expressed as a Trapezoidal Number in exactly one way?

EXPLORATION: Can you find a way to predict how many ways a number can be expressed as a Trapezoidal Number?

Puzzle of the Week

Trapezoidal Numbers – 4 – Notes

THE CHALLENGE & EXPLORATION: To get a feel for this, list out all the ways to write numbers as a sum of consecutive numbers. Writing out lots of examples is often a good way to look for patterns and get ideas.

- 1: No
- 2: No
- 3: 1 + 2
- 4: No
- 5: 2 + 3
- 6: 1 + 2 + 3
- 7: 3 + 4
- 8: No
- 9: 4 + 5; 2 + 3 + 4
- 10: 1 + 2 + 3 + 4
- 11: 5 + 6
- 12: 3 + 4 + 5
- 13: 6 + 7
- 14: 2 + 3 + 4 + 5
- 15: 7 + 8; 4 + 5 + 6; 1 + 2 + 3 + 4 + 5
- 16: No
- 17: 8 + 9
- 18: 5 + 6 + 7; 3 + 4 + 5 + 6
- 19: 9 + 10
- 20: 2 + 3 + 4 + 5 + 6

There is a pattern that is easy to see, and other patterns that are trickier.

1, 2, 4, 8, and 16 cannot be done. So it is reasonable to hypothesize that powers of 2 are not Trapezoidal Numbers.

9 has two ways, 15 has 3 ways, and 18 has two. For each of those, the number of ways is equal to the number of odd numbers that evenly divide the number! All these ideas lead to the following conjecture:

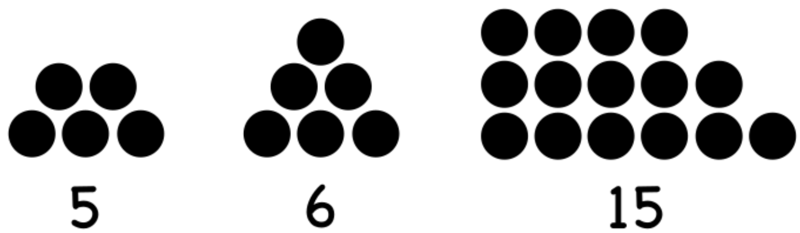
<u>Conjecture</u>: The number of ways to write a number as a sum of consecutive numbers is equal to the number of odd divisors larger than 1 of the number.

This is enough for now. If any of your students saw these patterns, that is wonderful! We will explore this a bit further in the Notes to the next Puzzle of the Week.

Puzzle of the Week

Trapezoidal Numbers – 5

Trapezoidal Numbers are the sum of two or more consecutive numbers. They deserve their name because you can make a trapezoid with that many dots, as pictured in the examples below. Note that having 1 dot on the top row is stretching the idea of being a trapezoid a bit, but it is allowed for these numbers.



THE CHALLENGE: Find one number between 100 and 200 that is <u>not</u> a Trapezoidal Number.

EXPLORATION: Trapezoidal Numbers are one example of what are called Figurate Numbers. A ***Figurate Number*** is a number that when that many dots are put in a special pattern, they make an interesting figure or shape. Triangular Numbers and Square Numbers are two other examples of Figurate Numbers. Can you think of other shapes of groups of dots that would deserve being called a Figurate Number?

Puzzle of the Week

Trapezoidal Numbers – 5 – Notes

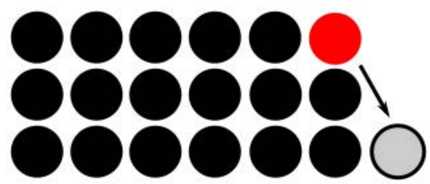
THE CHALLENGE: In the Notes of "Trapezoidal Numbers - 4" we looked at all the ways to write the numbers from 1 to 20 as sums of consecutive numbers. After looking at that data, we made the following conjecture

Conjecture: The number of ways to write a number as a sum of consecutive numbers is equal to the number of odd divisors larger than 1 of the number.

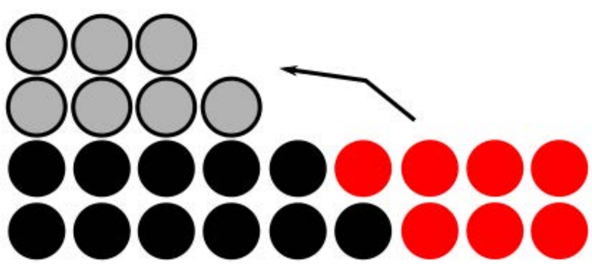
Let's look at why this conjecture is true. Suppose the number we are investigating is n, and that a = 2m + 1 is a divisor of n, so that a x b = n for some number b. There are two distinct ways of using this information to create a set of consecutive numbers that sum to n. One when

The number 18, with its two odd divisors 3 and 9, is useful in demonstrating the two ways of forming a set of consecutive numbers that sum to a number.

For 18's odd divisor 3: 3 x 6 = 18. Start by creating three rows of six dots. As shown in the illustration, leave the middle row intact, and remove a triangle of (red) dots from the upper rows (in this case, just one dot) and move that triangle to the ends of the lower rows. 18 = 5 + 6 + 7.



For 18's odd divisor 9: 9 x 2 = 18. Start by creating nine columns of two dots. As shown in the illustration, leaving the middle column intact, take a trapezoid of (red) dots from the right side and rotate it to be on top of the columns to the left of the middle column. 18 = 3 + 4 + 5 + 6.



The only numbers with no odd divisors are powers of 2. Because they have no odd divisors, the powers of 2 are the only numbers that are not Trapezoidal Numbers.

So, the answer to this Challenge is to find a power of 2 that is between 100 and 200. The only such number is 128, which is 2 to the seventh power.

EXPLORATION: Rather than repeat the material, please look up Figurate Numbers and Polygonal Numbers on the internet. Wikipedia has good introductory articles on both of them.

Puzzle of the Week

Turning the Tables

THE CHALLENGE: This started out as a standard multiplication table for the numbers 2 through 9. Then the rows and columns were all mixed up. Finally, most of the numbers have been removed. Put in all the missing numbers!

X								
2								
		40						
				49				
	20					36		
		72						
			9					12
					48			

EXPLORATION: Make one of these puzzles for someone else. How many numbers can you leave out and still have a puzzle that can be completely solved?

Puzzle of the Week

Turning the Tables – Notes

THE CHALLENGE: Start with the easiest rows and columns and work your way from there.

X	5	8	3	7		9		
2								
5		40						
7				49				
4	20					36		
9		72						
3			9					12
					48			

- 49 must be 7 x 7, so its row and column must be 7.
- 9 must be 3 x 3, so its row and column must be 3.
- 40 must be 5 x 8. 40 and 72 are in the same column, so 40 must be in column 8 and row 5
- 72 is in column 8, so it is in row 9.
- 20 must be 4 x 5. There is already a row for 5, so 20 must be in row 4 and column 5.
- 36 is in row 4, so it is in column 9.
- 12 is in row 3, so it is in column 4.

With that much information, the remaining numbers go very quickly.

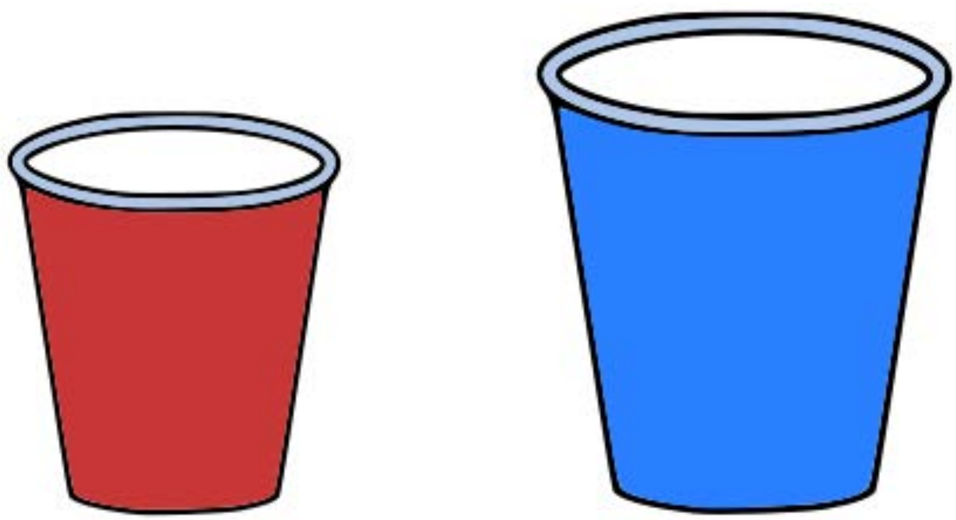
- 48 is 6 x 8 and there is already a column for 8, so 48 must be in column 6 and row 8.
- The only remaining row, the one at the top, must be for 6.
- The only remaining column, the one second from the far right, must be for 2.

Puzzle of the Week

Water Cups – 1

You have two unmarked water cups. One holds 3 ounces and the other 7 ounces. You also have a large supply of water. You can use these two cups to create amounts other than 3 ounces and 7 ounces. For example, create 4 ounces in the larger cup by filling the 7-ounce cup and then pouring 3 of its ounces into the smaller cup.

THE CHALLENGE: Describe the steps for putting 2 ounces in one of these cups.



3 Ounce 7 Ounce

EXPLORATION: Describe the steps to take to create any amount from 1 to 7 ounces. What is your general method? Experiment with what happens if your pair of cups have different sizes. For example, what happens if the sizes in ounces are 4 and 7, or 5 and 11?

Puzzle of the Week

Water Cups – 1 – Notes

THE CHALLENGE: There is a temptation with this puzzle to do a lot of disorganized experimenting that happens upon solutions accidentally. That is good fun and should not be discouraged. However, by simply refilling the larger cup, multiple times if need be, it is straightforward to achieve each amount.

Fill the 7-ounce cup. That achieves a **7-ounce** amount.

Next, pour out 3 ounces into the smaller cup. There are now **4 ounces** in the larger cup and **3 ounces** in the smaller.

Empty the 3-ounce cup and refill it from the larger cup. There is now **1 ounce** in the larger cup.

Empty the 3-ounce cup, pour 1 ounce from the larger cup into the smaller one, refill the larger cup, and fill the 3-ounce cup from the larger one. There are now **5 ounces** in the larger cup.

Empty the 3-ounce cup and refill it from the larger cup. There are now **2 ounces** in the larger cup.

Empty the 3-ounce cup, pour 2 ounces from the larger cup into the smaller one, refill the larger cup, and fill the 3-ounce cup from the larger one. There are now **6 ounces** in the larger cup.

Every amount has now occurred somewhere along the way. If you want all the amounts to occur in the larger cup, you can add one final additional step to this process (that is the same as all the others) to leave 3 ounces in the larger cup.

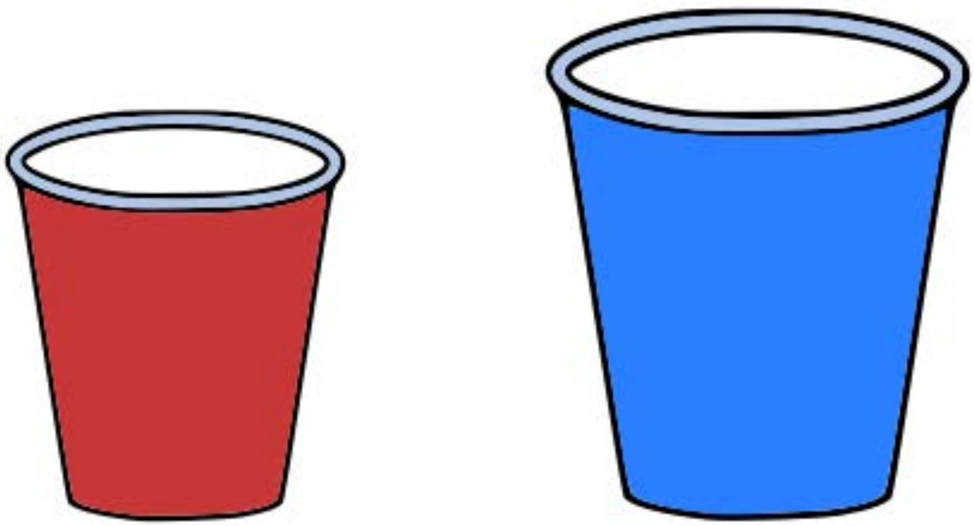
EXPLORATION: This simple process will always work as long as the two amounts are relatively prime, that is, they have no common factor larger than 1. So, if the amounts are 4 and 7, or 5 and 11, then all the amounts up to the size of the larger cup are easily obtainable.

Puzzle of the Week

Water Cups – 2

You have two unmarked water cups. One holds 9 ounces and the other 15 ounces. You also have a large supply of water. You can use these two cups to create amounts other than 9 ounces and 15 ounces. For example, create 6 ounces in the larger cup by filling the 15-ounce cup and then pouring 9 of its ounces into the smaller cup.

THE CHALLENGE: Find all the amounts that you can create using these two cups.



9 Ounce 15 Ounce

EXPLORATION: Investigate other pairs of water cups that involve two numbers with a common divisor greater than 1. What patterns do you notice?

Puzzle of the Week

Water Cups – 2 – Notes

THE CHALLENGE & EXPLORATION: The two numbers, 9 and 15, have a greatest common divisor of 3. Whatever the greatest common divisor is, you will only be able to produce answers that are a multiple of their greatest common divisor.

The easiest way to work with this problem is to imagine creating a new unit. In this case, let's call that unit ThreeOunce, and it will equal 3 ounces. So we have one cup that holds 3 ThreeOunces and the other holds 5 ThreeOunces. Now, the analysis proceeds exactly as it did in "Water Cups - 1" with a 3-unit and a 5-unit cup.

Fill the **5-unit** cup.

Pour **3 units** from the larger cup into the smaller cup, leaving **2 units** in the larger cup.

Empty the smaller cup, pour the 2 units into the smaller cup from the larger cup, refill the larger cup, and fill the smaller cup. That leaves **4 units** in the larger cup.

Empty the smaller cup and pour 3 units from the larger cup into the smaller cup. That leaves **1 unit** in the larger cup.

So, we have a method that produces 1 through 5 units, as expected.

This translates into being able to produce 3 ounces, 6 ounces, 9 ounces, 12 ounces, and 15 ounces.

In general, we will be able to produce every multiple of the greatest common divisor up to the size of the bigger cup.